Integral calculus: solved exercises

Exercise. Compute the following indefinite integrals:

$(a)\quad \displaystyle\int \frac{1+\cos x}{x+\sin x}\,dx \qquad [\log|x+\sin x|+c, \quad c\in\mathbb{R}]$

$(b)\quad \displaystyle\int \frac{3x+2}{x^2+1}\,dx \qquad \left[\frac{3}{2}\log(x^2+1)+2\arctan x+c, \quad c\in\mathbb{R}\right]$

$(c)\quad \displaystyle\int \frac{dx}{\sin^2 x\,\cos^2 x}. \qquad [\tan x-\cot x+c, \quad c\in\mathbb{R}]$

Solution

(a) Let us consider the indefinite integral

$$\int \frac{1+\cos x}{x+\sin x}\,dx.$$

Since $1+\cos x$ is the derivative of $x+\sin x$, we have that

$$\int \frac{1+\cos x}{x+\sin x}\,dx = \log|x+\sin x|+c\,, \qquad c\in\mathbb{R}.$$

(b) Let us consider the indefinite integral

$$\int \frac{3x+2}{x^2+1}\,dx.$$

We have that

$$\int \frac{3x+2}{x^2+1}\,dx = \int\left(\frac{3x}{x^2+1}+\frac{2}{x^2+1}\right)dx = \frac{3}{2}\int\frac{2x}{x^2+1}\,dx + 2\int\frac{1}{x^2+1}\,dx =$$

$$= \frac{3}{2}\log(x^2+1)+2\arctan x+c, \qquad c\in\mathbb{R}.$$

(c) Let us consider the indefinite integral

$$\int \frac{dx}{\sin^2 x\,\cos^2 x}.$$

Since $\sin^2 x+\cos^2 x=1$, we have that

$$\int \frac{1}{\sin^2 x\,\cos^2 x}\,dx = \int \frac{\sin^2 x+\cos^2 x}{\sin^2 x\,\cos^2 x}\,dx = \int\frac{1}{\cos^2 x}\,dx + \int\frac{1}{\sin^2 x}\,dx =$$

$$= \tan x-\cot x+c, \quad c\in\mathbb{R}.$$

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1. Integrating by parts

Exercise. Compute the following indefinite integrals, using integration by parts:

(a) $\displaystyle\int \arcsin x \, dx$ $\qquad\left[x \arcsin x + \sqrt{1-x^2} + c, \quad c \in \mathbb{R}\right]$

(b) $\displaystyle\int x^2 \log^2 x \, dx$ $\qquad\left[\frac{1}{3}x^3\left(\log^2 x - \frac{2}{3}\log x + \frac{2}{9}\right) + c, \quad c \in \mathbb{R}\right]$

(c) $\displaystyle\int x^3 \sqrt{2-x^2} \, dx.$ $\qquad\left[-\frac{1}{3}x^2\,(2-x^2)^{\frac{3}{2}} - \frac{2}{15}(2-x^2)^{\frac{5}{2}} + c, \quad c \in \mathbb{R}\right]$

Solution

(a) Let us consider the indefinite integral

$$\int \arcsin x \, dx.$$

Integrating by parts we have that

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx = x \arcsin x + \sqrt{1-x^2} + c, \qquad c \in \mathbb{R}.$$

(b) Let us consider the indefinite integral

$$\int (x \, \log x)^2 \, dx = \int x^2 \log^2 x \, dx.$$

Integrating twice by parts we have that

$$\begin{aligned}\int x^2 \log^2 x \, dx &= \frac{1}{3}x^3 \log^2 x - \frac{2}{3}\int x^2 \log x \, dx = \\ &= \frac{1}{3}x^3 \log^2 x - \frac{2}{9}x^3 \log x + \frac{2}{9}\int x^2 \, dx = \\ &= \frac{1}{3}x^3 \log^2 x - \frac{2}{9}x^3 \log x + \frac{2}{27}x^3 + c = \\ &= \frac{1}{3}x^3\left(\log^2 x - \frac{2}{3}\log x + \frac{2}{9}\right) + c, \qquad c \in \mathbb{R}.\end{aligned}$$

(c) Let us consider the indefinite integral

$$\int x^3 \sqrt{2-x^2} \, dx = \int x^2 \left(x\sqrt{2-x^2}\right) \, dx \, .$$

Integrating by parts we have that

$$\begin{aligned}\int x^2 \left(x\sqrt{2-x^2}\right) \, dx &= -\frac{1}{3}x^2\,(2-x^2)^{\frac{3}{2}} + \frac{2}{3}\int x\,(2-x^2)^{\frac{3}{2}} \, dx = \\ &= -\frac{1}{3}x^2\,(2-x^2)^{\frac{3}{2}} - \frac{2}{15}(2-x^2)^{\frac{5}{2}} + c, \qquad c \in \mathbb{R}.\end{aligned}$$

1. Integrating by substitution

Exercise. Compute the following indefinite integrals by substitution:

$(a)\quad \displaystyle\int \frac{dx}{x\,\log^3 x}$ $\qquad\left[-\dfrac{1}{2\log^2 x}+c,\quad c\in\mathbb{R}\right]$

$(b)\quad \displaystyle\int \frac{\sin 2x}{1+\sin^2 x}\,dx$ $\qquad\left[\log\left(1+\sin^2 x\right)+c,\quad c\in\mathbb{R}\right]$

$(c)\quad \displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}.$ $\qquad\left[2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\log\left(\sqrt[6]{x}+1\right)+c,\quad c\in\mathbb{R}\right]$

Solution

(a) Let us consider the indefinite integral
$$\int \frac{dx}{x\,\log^3 x}.$$
Setting $t=\log x$ we have that $dt=\frac{1}{x}\,dx$. Hence
$$\int \frac{dx}{x\,\log^3 x}=\int \frac{1}{t^3}\,dt=-\frac{1}{2t^2}+c=-\frac{1}{2\log^2 x}+c,\qquad c\in\mathbb{R}.$$

(b) Let us consider the indefinite integral
$$\int \frac{\sin 2x}{1+\sin^2 x}\,dx=\int \frac{2\sin x\cos x}{1+\sin^2 x}\,dx.$$
Setting $t=\sin x$ we have that $dt=\cos x\,dx$. Hence
$$\int \frac{2\sin x\cos x}{1+\sin^2 x}\,dx=\int \frac{2t}{1+t^2}\,dt=\log\left(1+t^2\right)+c=\log\left(1+\sin^2 x\right)+c,\quad c\in\mathbb{R}.$$

(c) Let us consider the indefinite integral
$$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}.$$
Setting $x=t^6$, we have that $dx=6t^5\,dt$. Hence
$$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}=6\int \frac{t^3}{t+1}\,dt=6\int\left(t^2-t+1-\frac{1}{t+1}\right)\,dt=$$
$$=2t^3-3t^2+6t-6\log|t+1|+c=2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\log\left(\sqrt[6]{x}+1\right)+c,\qquad c\in\mathbb{R}.$$

1. Integrating rational maps

Exercise. Compute the following indefinite integrals of rational maps:

$(a)\quad \displaystyle\int \frac{x+1}{x\,(1+x^2)}\,dx$ $\qquad \left[\log \frac{|x|}{\sqrt{1+x^2}} + \arctan x + c, \quad c \in \mathbb{R}\right]$

$(b)\quad \displaystyle\int \frac{1}{x^3\,(1+x^2)}\,dx$ $\qquad \left[\log \frac{\sqrt{1+x^2}}{|x|} - \frac{1}{2x^2} + c, \quad c \in \mathbb{R}\right]$

$(c)\quad \displaystyle\int \frac{x^3+x^2-x}{x^2+x-6}\,dx$ $\qquad \left[\frac{1}{2}x^2 + 2\log|x-2| + 3\log|x+3| + c, \quad c \in \mathbb{R}\right]$

$(d)\quad \displaystyle\int \frac{dx}{x(x^2+2x+3)}$ $\qquad \left[\log \sqrt[6]{\frac{x^2}{x^2+2x+3}} - \frac{\sqrt{2}}{6}\arctan\frac{x+1}{\sqrt{2}} + c, \quad c \in \mathbb{R}\right]$

$(e)\quad \displaystyle\int \frac{x^2-10x+10}{x^3+2x^2+5x}\,dx.$ $\qquad \left[\log \frac{x^2}{\sqrt{x^2+2x+5}} - \frac{13}{2}\arctan\frac{x+1}{2} + c, \quad c \in \mathbb{R}\right]$

Solution

(a) Let us consider the indefinite integral

$$\int \frac{x+1}{x\,(1+x^2)}\,dx.$$

We have that

$$\frac{x+1}{x\,(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} = \frac{(A+B)x^2+Cx+A}{x(1+x^2)} \quad \Longrightarrow \quad \begin{cases} A = C = 1 \\ B = -1. \end{cases}$$

Hence

$$\begin{aligned}
\int \frac{x+1}{x\,(1+x^2)}\,dx \quad &= \int \left(\frac{1}{x} + \frac{1-x}{1+x^2}\right)dx = \\
&= \int \frac{1}{x}\,dx + \int \frac{1}{1+x^2}\,dx - \int \frac{x}{1+x^2}\,dx = \\
&= \log|x| + \arctan x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx = \\
&= \log|x| + \arctan x - \frac{1}{2}\log(1+x^2) + c = \\
&= \log\frac{|x|}{\sqrt{1+x^2}} + \arctan x + c, \qquad c \in \mathbb{R}.
\end{aligned}$$

(b) Let us consider the indefinite integral

$$\int \frac{1}{x^3\,(1+x^2)}\,dx.$$

We have that

$$\frac{1}{x^3\,(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} + \frac{d}{dx}\left(\frac{Dx+E}{x^2}\right) = \frac{A}{x} + \frac{Bx+C}{1+x^2} - \frac{Dx+2E}{x^3} =$$

$$= \frac{(A+B)x^4 + (C-D)x^3 + (A-2E)x^2 - Dx - 2E}{x^3(1+x^2)} \quad \Longrightarrow \quad \begin{cases} A = -1 \\ B = 1 \\ C = D = 0 \\ E = -\dfrac{1}{2}. \end{cases}$$

Hence

$$\begin{aligned}
\int \frac{1}{x^3\,(1+x^2)}\,dx &= \int \left[-\frac{1}{x} + \frac{x}{1+x^2} + \frac{d}{dx}\left(-\frac{1}{2x^2}\right)\right] dx = \\
&= -\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{2x}{1+x^2}\,dx + \int \frac{d}{dx}\left(-\frac{1}{2x^2}\right) dx = \\
&= -\log|x| + \frac{1}{2}\log\left(1+x^2\right) - \frac{1}{2x^2} + c = \\
&= \log\frac{\sqrt{1+x^2}}{|x|} - \frac{1}{2x^2} + c, \qquad c \in \mathbb{R}.
\end{aligned}$$

(c) Let us consider the indefinite integral

$$\int \frac{x^3+x^2-x}{x^2+x-6}\,dx.$$

Dividing x^3+x^2-x by x^2+x-6, we have that

$$\frac{x^3+x^2-x}{x^2+x-6} = x + \frac{5x}{x^2+x-6}.$$

Hence

$$\int \frac{x^3+x^2-x}{x^2+x-6}\,dx = \int \left(x + \frac{5x}{x^2+x-6}\right) dx = \frac{1}{2}x^2 + \int \frac{5x}{(x-2)(x+3)}\,dx.$$

We have that

$$\frac{5x}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3} = \frac{(A+B)x + 3A - 2B}{(x-2)(x+3)} \quad \Longrightarrow \quad \begin{cases} A = 2 \\ B = 3. \end{cases}$$

Hence

$$\begin{aligned}
\int \frac{x^3+x^2-x}{x^2+x-6}\,dx &= \frac{1}{2}x^2 + \int \frac{5x}{(x-2)(x+3)}\,dx = \\
&= \frac{1}{2}x^2 + \int \left(\frac{2}{x-2} + \frac{3}{x+3}\right) dx = \\
&= \frac{1}{2}x^2 + 2\int \frac{1}{x-2}\,dx + 3\int \frac{1}{x+3}\,dx = \\
&= \frac{1}{2}x^2 + 2\log|x-2| + 3\log|x+3| + c, \qquad c \in \mathbb{R}.
\end{aligned}$$

(d) Let us consider the indefinite integral

$$\int \frac{dx}{x(x^2+2x+3)}.$$

We have that

$$\frac{1}{x(x^2+2x+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+3} =$$

$$= \frac{(A+B)x^2+(2A+C)x+3A}{x(x^2+2x+3)} \implies \begin{cases} A = \frac{1}{3} \\ B = -\frac{1}{3} \\ C = -\frac{2}{3}. \end{cases}$$

Hanec we have that

$$\int \frac{dx}{x(x^2+2x+3)} = \int \left(\frac{1}{3x} - \frac{1}{3}\frac{x+2}{x^2+2x+3} \right) dx =$$

$$= \frac{1}{3}\log|x| - \frac{1}{6}\int \frac{2x+2}{x^2+2x+3}\,dx - \frac{1}{3}\int \frac{1}{x^2+2x+3}\,dx =$$

$$= \frac{1}{3}\log|x| - \frac{1}{6}\log(x^2+2x+3) - \frac{1}{3}\int \frac{1}{x^2+2x+3}\,dx =$$

being $\quad x^2+2x+3 = (x+1)^2+2 = 2\left[\left(\frac{x+1}{\sqrt{2}}\right)^2+1\right]$, we have that

$$= \frac{1}{3}\log|x| - \frac{1}{6}\log(x^2+2x+3) - \frac{1}{6}\int \frac{1}{\left[\left(\frac{x+1}{\sqrt{2}}\right)^2+1\right]}\,dx =$$

posto $t = \frac{x+1}{\sqrt{2}}$, we have that $dt = \frac{1}{\sqrt{2}}\,dx$, hence

$$= \frac{1}{3}\log|x| - \frac{1}{6}\log(x^2+2x+3) - \frac{\sqrt{2}}{6}\int \frac{1}{t^2+1}\,dt =$$

$$= \frac{1}{3}\log|x| - \frac{1}{6}\log(x^2+2x+3) - \frac{\sqrt{2}}{6}\arctan t + c =$$

$$= \frac{1}{3}\log|x| - \frac{1}{6}\log(x^2+2x+3) - \frac{\sqrt{2}}{6}\arctan\frac{x+1}{\sqrt{2}} + c =$$

$$= \log\sqrt[6]{\frac{x^2}{x^2+2x+3}} - \frac{\sqrt{2}}{6}\arctan\frac{x+1}{\sqrt{2}} + c\,, \qquad c \in \mathbb{R}\,.$$

(e) Let us consider the indefinite integral

$$\int \frac{x^2-10x+10}{x^3+2x^2+5x}\,dx = \int \frac{x^2-10x+10}{x(x^2+2x+5)}\,dx\,.$$

We have that

$$\frac{x^2-10x+10}{x(x^2+2x+5)} = \frac{A}{x}+\frac{Bx+C}{x^2+2x+5} =$$

$$= \frac{(A+B)x^2+(2A+C)x+5A}{x(x^2+2x+5)} \implies \begin{cases} A=2 \\ B=-1 \\ C=-14. \end{cases}$$

Hence

$$\int \frac{x^2-10x+10}{x(x^2+2x+5)}\,dx = \int\left(\frac{2}{x}-\frac{x+14}{x^2+2x+5}\right)dx =$$

$$= \int\left(\frac{2}{x}-\frac{x+1}{x^2+2x+5}-\frac{13}{x^2+2x+5}\right)dx =$$

$$= 2\log|x| - \frac{1}{2}\int\frac{2x+2}{x^2+2x+5}\,dx - 13\int\frac{1}{x^2+2x+5}\,dx =$$

$$= 2\log|x| - \frac{1}{2}\log(x^2+2x+5) - 13\int\frac{1}{x^2+2x+5}\,dx\,.$$

Since

$$x^2+2x+5 = (x+1)^2+4 = 4\left[\left(\frac{x+1}{2}\right)^2+1\right],$$

we have that

$$\int\frac{1}{x^2+2x+5}\,dx = \int\frac{1}{4\left[\left(\frac{x+1}{2}\right)^2+1\right]}\,dx = \frac{1}{2}\int\frac{\frac{1}{2}}{\left(\frac{x+1}{2}\right)^2+1}\,dx$$

$$= \frac{1}{2}\arctan\frac{x+1}{2}+c\,, \qquad c\in\mathbb{R}.$$

Hence

$$\int \frac{x^2-10x+10}{x(x^2+2x+5)}\,dx = 2\log|x| - \frac{1}{2}\log(x^2+2x+5) - 13\int\frac{1}{x^2+2x+5}\,dx =$$

$$= 2\log|x| - \frac{1}{2}\log(x^2+2x+5) - \frac{13}{2}\arctan\frac{x+1}{2}+c =$$

$$= \log\frac{x^2}{\sqrt{x^2+2x+5}} - \frac{13}{2}\arctan\frac{x+1}{2}+c\,, \qquad c\in\mathbb{R}\,.$$

Substitutions of special type

Exercise. Compute the following indefinite integrals by substitutions:

(a) $\displaystyle\int\frac{1}{\sin x}\,dx$ $\qquad\left[\log\left|\tan\frac{x}{2}\right|+c, \quad c\in\mathbb{R}\right]$

(b) $\displaystyle\int\frac{dx}{x^2\sqrt{4+x^2}}$ $\qquad\left[-\frac{1}{x^2+x\sqrt{x^2+4}}+c, \quad c\in\mathbb{R}\right]$

(c) $\displaystyle\int \frac{dx}{\sqrt{1+2x-x^2}}$. $\left[\arcsin\dfrac{x-1}{\sqrt{2}}+c, \quad c\in\mathbb{R}\right]$

Solution

(a) Let us consider the indefinite integral

$$\int \frac{1}{\sin x}\,dx.$$

Setting $t=\tan\frac{x}{2}$ we have that $dx=\frac{2}{1+t^2}\,dt$. Since $\sin x=\frac{2t}{1+t^2}$, we have that

$$\int \frac{1}{\sin x}\,dx=\int \frac{1}{t}\,dt=\log|t|+c=\log\left|\tan\frac{x}{2}\right|+c, \quad c\in\mathbb{R}.$$

(b) Let us consider the indefinite integral

$$\int \frac{dx}{x^2\sqrt{4+x^2}}.$$

Setting $x=2\sinh t$, i.e. $t=\operatorname{settsinh}\frac{x}{2}=\log\left(\frac{x}{2}+\frac{1}{2}\sqrt{x^2+4}\right)$, from which it follows $dx=2\cosh t\,dt$, hence

$$\int \frac{dx}{x^2\sqrt{4+x^2}}=\frac{1}{4}\int\frac{1}{\sinh^2 t}\,dt=\int\frac{e^{2t}}{(e^{2t}-1)^2}\,dt.$$

Setting $z=e^t$, cioè $z=\frac{x}{2}+\frac{1}{2}\sqrt{x^2+4}$, from which it follows $dz=e^t\,dt$, we have that

$$\int \frac{dx}{x^2\sqrt{4+x^2}}=\int\frac{e^{2t}}{(e^{2t}-1)^2}\,dt=\int\frac{z}{(z^2-1)^2}\,dz=-\frac{1}{2}\frac{1}{z^2-1}+c=$$

$$=-\frac{1}{x^2+x\sqrt{x^2+4}}+c, \qquad c\in\mathbb{R}.$$

(c) Let us consider the indefinite integral

$$\int \frac{dx}{\sqrt{1+2x-x^2}}=\int\frac{dx}{\sqrt{2-(x-1)^2}}.$$

Setting $x-1=\sqrt{2}\sin t$, for all $t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we have that $t=\arcsin\frac{x-1}{\sqrt{2}}$, $\cos t=\sqrt{1-\sin^2 t}$ e $dx=\sqrt{2}\cos t\,dt$. Hence

$$\int\frac{dx}{\sqrt{2-(x-1)^2}}=\int dt=t+c=\arcsin\frac{x-1}{\sqrt{2}}+c, \qquad c\in\mathbb{R}.$$

Integrating piecewise defined functions

Exercise. Compute the following indefinite integrals of piecewise defined functions:

$(a)\quad f(x)=\begin{cases} xe^x & \text{if } x\le 0\\ \sin x & \text{if } x>0\end{cases}$ $\left[\begin{cases} e^x(x-1)+c & \text{if } x\le 0\\ -\cos x+c & \text{if } x>0,\end{cases}\quad c\in\mathbb{R}\right]$

$(b)\quad f(x)=\begin{cases} -x^3\,\sin(\pi+\pi x^2) & \text{if } x\le 1\\ x^2-8x+7 & \text{if } x>1.\end{cases}$

$$\left[\begin{cases} -\dfrac{1}{2\pi}x^2\cos(\pi x^2)+\dfrac{1}{2\pi^2}\sin(\pi x^2)+c & \text{if } x\le 1\\ \dfrac{1}{3}x^3-4x^2+7x+c+\dfrac{1}{2\pi}-\dfrac{10}{3} & \text{if } x>1,\end{cases}\quad c\in\mathbb{R}\right]$$

Solution

(a) Let us consider the function

$$f(x)=\begin{cases} xe^x & \text{if } x\le 0\\ \sin x & \text{if } x>0.\end{cases}$$

Let us find an arbitrary primitive function F of f on $\mathbb{R}$. We have that

$$\int xe^x\,dx = xe^x-\int e^x\,dx = xe^x-e^x+c_1=e^x(x-1)+c_1,\qquad c_1\in\mathbb{R},$$

$$\int \sin x\,dx = -\cos x+c_2,\qquad c_2\in\mathbb{R}.$$

Hence

$$F(x)=\begin{cases} e^x(x-1)+c_1 & \text{if } x\le 0\\ -\cos x+c_2 & \text{if } x>0,\end{cases}$$

where $c_1, c_2\in\mathbb{R}$ are such that the primitive function F is continuous at 0. Hence

$$F(0)=\lim_{x\to 0^+}F(x).$$

Since

$$F(0)=c_1-1\,,\qquad \lim_{x\to 0^+}F(x)=c_2-1,$$

we have that $c_1=c_2$. So, setting $c=c_1$, we have that any primitive function of f is of the form

$$F(x)=\begin{cases} e^x(x-1)+c & \text{if } x\le 0\\ -\cos x+c & \text{if } x>0,\end{cases}\qquad c\in\mathbb{R}.$$

(b) Let us consider the function

$$f(x)=\begin{cases} -x^3\,\sin(\pi+\pi x^2) & \text{if } x\le 1\\ x^2-8x+7 & \text{if } x>1.\end{cases}$$

Let us find an arbitrary primitive function F of f on $\mathbb{R}$. We have that

$$-\int x^3\sin(\pi+\pi x^2)\,dx = \int x^3\sin(\pi x^2)\,dx=\int x\left(x^2\sin(\pi x^2)\right)\,dx=$$

integrating by parts we have

$$= -\frac{1}{2\pi}x^2\cos(\pi x^2) + \frac{1}{\pi}\int x\cos(\pi x^2)\,dx =$$

$$= -\frac{1}{2\pi}x^2\cos(\pi x^2) + \frac{1}{2\pi^2}\sin(\pi x^2) + c_1, \qquad c_1 \in \mathbb{R},$$

$$\int (x^2 - 8x + 7)\,dx = \frac{1}{3}x^3 - 4x^2 + 7x + c_2, \qquad c_2 \in \mathbb{R}.$$

Hence

$$F(x) = \begin{cases} -\frac{1}{2\pi}x^2\cos(\pi x^2) + \frac{1}{2\pi^2}\sin(\pi x^2) + c_1 & \text{if } x \le 1 \\ \frac{1}{3}x^3 - 4x^2 + 7x + c_2 & \text{if } x > 1 \end{cases},$$

where $c_1, c_2 \in \mathbb{R}$ are such that the arbitrary primitive function F is continuous at 1. Hence

$$F(1) = \lim_{x\to 1^+} F(x).$$

Since

$$F(1) = c_1 + \frac{1}{2\pi}, \qquad \lim_{x\to 1^+} F(x) = c_2 + \frac{10}{3},$$

we have that

$$c_2 = c_1 + \frac{1}{2\pi} - \frac{10}{3}.$$

Hence, setting $c = c_1$, we have that any primitive function of f is of the form

$$F(x) = \begin{cases} -\frac{1}{2\pi}x^2\cos(\pi x^2) + \frac{1}{2\pi^2}\sin(\pi x^2) + c & \text{if } x \le 1 \\ \frac{1}{3}x^3 - 4x^2 + 7x + c + \frac{1}{2\pi} - \frac{10}{3} & \text{if } x > 1, \end{cases} \qquad c \in \mathbb{R}.$$

Definite integrals

Exercise. Compute the following definite integrals:

(a) $\displaystyle\int_0^{\pi} |6x - \pi|\sin x\,dx$ $[6\pi - 6]$

(b) $\displaystyle\int_{-\frac{\pi}{2}}^{0} \frac{2\sin^2 x + 3\sin x + 3}{(\sin x - 1)(\sin^2 x + 3)}\cos x\,dx$ $\left[\frac{\sqrt{3}}{6}\pi - 2\log 2\right]$

(c) $\displaystyle\int_e^{e^{\frac{3}{2}}} \frac{1}{x\left(1 - \sqrt{\log x - 1}\right)}\,dx.$ $\left[-\sqrt{2} - 2\log\left(1 - \frac{\sqrt{2}}{2}\right)\right]$

Solution

(*a*) Let us consider the definite integral

$$\int_0^{\pi} |6x-\pi| \sin x \, dx.$$

We have that

$$\int_0^{\pi} |6x-\pi| \sin x \, dx = -\int_0^{\frac{\pi}{6}} (6x-\pi) \sin x \, dx + \int_{\frac{\pi}{6}}^{\pi} (6x-\pi) \sin x \, dx =$$

integrating by parts

$$= \Big[(6x-\pi)\cos x\Big]_0^{\frac{\pi}{6}} - 6\int_0^{\frac{\pi}{6}} \cos x \, dx + \Big[-(6x-\pi)\cos x\Big]_{\frac{\pi}{6}}^{\pi} + 6\int_{\frac{\pi}{6}}^{\pi} \cos x \, dx =$$

$$= \pi - 6\Big[\sin x\Big]_0^{\frac{\pi}{6}} + 5\pi + 6\Big[\sin x\Big]_{\frac{\pi}{6}}^{\pi} = 6\pi - 6.$$

(*b*) Let us consider the definite integral

$$\int_{-\frac{\pi}{2}}^{0} \frac{2\sin^2 x + 3\sin x + 3}{(\sin x - 1)(\sin^2 x + 3)} \cos x \, dx.$$

Setting $t = \sin x$, from which it follows $dt = \cos x \, dx$, we have that

$$\int_{-\frac{\pi}{2}}^{0} \frac{2\sin^2 x + 3\sin x + 3}{(\sin x - 1)(\sin^2 x + 3)} \cos x \, dx = \int_{-1}^{0} \frac{2t^2+3t+3}{(t-1)(t^2+3)} \, dt.$$

We have that

$$\frac{2t^2+3t+3}{(t-1)(t^2+3)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+3} = \frac{(A+B)t^2 + (-B+C)t + 3A - C}{(t-1)(t^2+3)}$$

$$\Longrightarrow \quad \begin{cases} A = 2 \\ B = 0 \\ C = 3. \end{cases}$$

Hence we have that

$$\int_{-1}^{0} \frac{2t^2+3t+3}{(t-1)(t^2+3)} \, dt = \int_{-1}^{0} \left(\frac{2}{t-1} + \frac{3}{t^2+3} \right) dt =$$

$$= 2\Big[\log|t-1|\Big]_{-1}^{0} + \sqrt{3}\int_{-1}^{0} \frac{\frac{1}{\sqrt{3}}}{\left(\frac{t}{\sqrt{3}}\right)^2 + 1} \, dt =$$

$$= -2\log 2 + \sqrt{3}\left[\arctan \frac{t}{\sqrt{3}}\right]_{-1}^{0} = \frac{\sqrt{3}}{6}\pi - 2\log 2.$$

(*c*) Let us consider the definite integral

$$\int_e^{e^{\frac{3}{2}}} \frac{1}{x\left(1 - \sqrt{\log x - 1}\right)} \, dx.$$

Setting $t = \log x$, da cui $dt = \frac{1}{x}dx$, we have that

$$\int_e^{e^{\frac{3}{2}}} \frac{1}{x\left(1 - \sqrt{\log x - 1}\right)}\, dx = \int_1^{\frac{3}{2}} \frac{1}{1 - \sqrt{t-1}}\, dt\,.$$

Setting $y = \sqrt{t-1}$, da cui $t = y^2 + 1$ and hence $dt = 2ydy$, we have that

$$\int_1^{\frac{3}{2}} \frac{1}{1-\sqrt{t-1}}\, dt = 2\int_0^{\frac{\sqrt{2}}{2}} \frac{y}{1-y}\, dy = 2\int_0^{\frac{\sqrt{2}}{2}} \left(1 - \frac{1}{1-y}\right) dy =$$

$$= 2\Big[-y - \log|1-y|\Big]_0^{\frac{\sqrt{2}}{2}} = -\sqrt{2} - 2\log\left(1 - \frac{\sqrt{2}}{2}\right).$$

Other exercises

Exercise 1. Write the McLaurin expansion of order 6 of the function

$$f(x) = \arctan x \cdot \int_0^x e^{-t^2}\, dt.$$

Solution

It is known that if g is a continuous function defined in a neighbourhood of 0 and if $\alpha > 0$, then

$$g(x) = o\left(|x|^\alpha\right), \quad x \to 0 \quad \Longrightarrow \quad \int_0^x g(t)\, dt = o\left(|x|^{\alpha+1}\right), \quad x \to 0.$$

Hence, using the McLaurin expansions of functions $\arctan x$ and e^s we get

$$f(x) = \arctan x \cdot \int_0^x e^{-t^2}\, dt =$$

$$= \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + o\left(x^5\right)\right) \cdot \int_0^x \left(1 - t^2 + \frac{1}{2}t^4 + o\left(t^4\right)\right) dt =$$

$$= \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + o\left(x^5\right)\right) \cdot \left(\left[t - \frac{1}{3}t^3 + \frac{1}{10}t^5\right]_0^x + o\left(x^5\right)\right) =$$

$$= \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + o\left(x^5\right)\right) \cdot \left(x - \frac{1}{3}x^3 + \frac{1}{10}x^5 + o\left(x^5\right)\right) =$$

$$= x^2 - \frac{2}{3}x^4 + \frac{37}{90}x^6 + o\left(x^6\right), \quad x \to 0.$$

It follows that the McLaurin expansion of order 6 of f is

$$f(x) = x^2 - \frac{2}{3}x^4 + \frac{37}{90}x^6 + o\left(x^6\right), \quad x \to 0.$$

Exercise 2. Write the McLaurin expansion of order 9 of the primitive function of

$$f(x) = \cos 2x^2$$

which takes the value 0 at $x = 0$.

Solution

Since f is continuous, By the Fundamental Theorem of Integral Calculus, the function $F(x) = \int_0^x f(t)\,dt = \int_0^x \cos 2t^2\,dt$ is the primitive function of f which takes the value 0 at $x = 0$. Moreover, it is well-known that if g is a continuous function defined in a neighbourhood of 0 and if $\alpha > 0$, then

$$g(x) = o\left(|x|^\alpha\right), \quad x \to 0 \quad \Longrightarrow \quad \int_0^x g(t)\,dt = o\left(|x|^{\alpha+1}\right), \quad x \to 0.$$

Hence, using the McLaurin expansion of $\cos s$ we get

$$F(x) = \int_0^x \cos 2t^2\,dt = \int_0^x \left(1 - 2t^4 + \frac{2}{3}t^8 + o\left(t^8\right)\right) dt =$$

$$= \left(\left[t - \frac{2}{5}t^5 + \frac{2}{27}t^9\right]_0^x + o\left(x^9\right)\right) = x - \frac{2}{5}x^5 + \frac{2}{27}x^9 + o\left(x^9\right), \quad x \to 0.$$

It follows that the McLaurin expansion of order 6 of F is

$$F(x) = x - \frac{2}{5}x^5 + \frac{2}{27}x^9 + o\left(x^9\right), \quad x \to 0.$$

Exercise 3. Compute the area of the following subsets of the plane:

(*a*) $A = \left\{(x, y) \in \mathbb{R}^2 : \ 1 \le x \le 2, \ 0 \le y \le \dfrac{1}{x\left(1 - \log^2 x\right)}\right\}$ $\left[\dfrac{1}{2}\log\left(\dfrac{1+\log 2}{1-\log 2}\right)\right]$

(*b*) $B = \left\{(x, y) \in \mathbb{R}^2 : \ -\sqrt{5} \le x \le -1, \ \dfrac{x}{x^2 + 2\sqrt{x^2-1}} \le y \le 0\right\}$ $\left[\log 3 - \dfrac{2}{3}\right]$

(*c*) $C = \left\{(x, y) \in \mathbb{R}^2 : \ 1 \le x \le e, \ \dfrac{\log x}{x\sqrt{4 + 3\log^2 x}} \le y \le x^2\right\}.$ $\left[\dfrac{1}{3}\left(e^3 + 1 - \sqrt{7}\right)\right]$

Solution

(*a*) Setting $f(x) = \frac{1}{x(1-\log^2 x)}$, let us note that as $1 \le x \le 2$ we have that $f(x) \ge 0$. Hence the area of A is given by

$$\text{Area}_A = \int_1^2 f(x)\,dx = \int_1^2 \frac{1}{x\left(1 - \log^2 x\right)}\,dx.$$

Setting $t = \log x$, so that $dt = \frac{1}{x}\,dx$, we have that

$$\int_1^2 \frac{1}{x\left(1-\log^2 x\right)}\,dx = \int_0^{\log 2} \frac{1}{1-t^2}\,dt = \frac{1}{2}\int_0^{\log 2}\left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt =$$

$$= \frac{1}{2}\Big[-\log|1-t| + \log|1+t|\Big]_0^{\log 2} = \frac{1}{2}\left[\log\left(\frac{1+t}{1-t}\right)\right]_0^{\log 2} = \frac{1}{2}\log\left(\frac{1+\log 2}{1-\log 2}\right).$$

Hence the area of A is $\text{Area}_A = \frac{1}{2}\log\left(\frac{1+\log 2}{1-\log 2}\right)$.

(*b*) Setting $f(x) = \frac{x}{x^2+2\sqrt{x^2-1}}$, let us note that per $-\sqrt{5} \leq x \leq -1$ we have that $f(x) \leq 0$. Hence the area of B is given by

$$\text{Area}_B = -\int_{-\sqrt{5}}^{-1} f(x)\,dx = -\int_{-\sqrt{5}}^{-1} \frac{x}{x^2+2\sqrt{x^2-1}}\,dx =$$

setting $t = \sqrt{x^2-1}$, and hence $x^2 = t^2+1$ and $xdx = t\,dt$,

$$= -\int_{-\sqrt{5}}^{-1} \frac{x}{x^2+2\sqrt{x^2-1}}\,dx = -\int_{2}^{0} \frac{t}{(t+1)^2}\,dt = -\int_{2}^{0} \left(\frac{1}{t+1} - \frac{1}{(t+1)^2}\right)dt =$$

$$= -\left[\log|t+1| + \frac{1}{(t+1)}\right]_2^0 = \log 3 - \frac{2}{3}.$$

Hence the area of B is $\text{Area}_B = \log 3 - \frac{2}{3}$.

(*c*) Setting $f(x) = \frac{\log x}{x\sqrt{4+3\log^2 x}}$, let us note that, as $1 \leq x \leq e$, we have that $f(x) \leq x^2$. In fact,

$$\frac{\log x}{x\sqrt{4+3\log^2 x}} \leq x^2 \quad \Longleftrightarrow \quad \frac{\log x}{\sqrt{4+3\log^2 x}} \leq x^3$$

and the functions $g(x) = \frac{\log x}{\sqrt{4+3\log^2 x}}$ e $h(x) = x^3$ are increasing in the interval $[1,e]$ with $0 \leq g(x) \leq \frac{1}{\sqrt{7}}$, $1 \leq h(x) \leq e^3$ for all $x \in [1,e]$. It follows that $g(x) \leq h(x)$ for all $x \in [1,e]$, i.e. $f(x) \leq x^2$ for all $x \in [1,e]$. So the are of C is given by

$$\text{Area}_C = \int_1^e \left(x^2 - \frac{\log x}{x\sqrt{4+3\log^2 x}}\right)dx = \left[\frac{1}{3}x^3\right]_1^e - \int_1^e \frac{\log x}{x\sqrt{4+3\log^2 x}}\,dx =$$

setting $t = \log x$ and $dt = \frac{1}{x}dx$,

$$= \frac{1}{3}\left(e^3-1\right) - \int_0^1 \frac{t}{\sqrt{4+3t^2}}\,dt = \frac{1}{3}\left(e^3-1\right) - \int_0^1 t(4+3t^2)^{-\frac{1}{2}}\,dt =$$

$$= \frac{1}{3}\left(e^3-1\right) - \left[\frac{1}{3}\sqrt{4+3t^2}\right]_0^1 = \frac{1}{3}\left(e^3+1-\sqrt{7}\right).$$

Hence, the area of C is $\text{Area}_C = \frac{1}{3}\left(e^3+1-\sqrt{7}\right)$.